Problem 2

2- **Convert the following Transfer Function to the Controllable Canonical Form state space representation:**

$$\frac{S^2 + 7S + 2}{S^3 + 9S^2 + 26S + 24}$$

solution

1 State-Space Representation in Canonical Forms

We here consider a system defined by

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u\,, \tag{1}$$

where u is the control input and y is the output. We can write this equation as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}\,. \tag{2}$$

Later, we shall present state-space representation of the system defined by (1) and (2) in controllable canonical form, observable canonical form, and diagonal canonical form.

1.1 Controllable Canonical Form

We consider the following state-space representation, being called a controllable canonical form, as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{3}$$

$$y = \begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \cdots & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \tag{4}$$

Note that the controllable canonical form is important in dicsussing the pole-placement approach to the control system design.

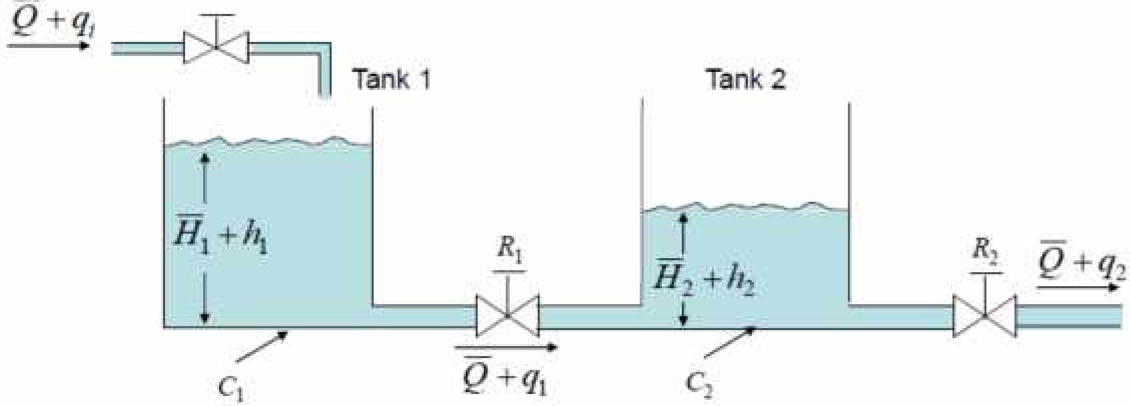
And calculate according to the problem ☺

Problem 3

3- Drive the mathematical model of the double tank system shown.

Liquid-level System

- Double-tank System:



Tank 1:

$$C_1\frac{dh_1}{dt}=q_i-q_1 \qquad (1)$$

$$q_1=\frac{h_1-h_2}{R_1} \qquad (2)$$

Tank 2:

$$C_2\frac{dh_2}{dt}=q_1-q_2 \qquad (3)$$

$$q_2=\frac{h_2}{R_2} \qquad (4)$$

Liquid-level System

- Transfer function of a double-tank system:

$$\frac{Q_2(s)}{Q_i(s)} = ?$$

$$C_1 \frac{dh_1}{dt} = q_i - q_1 \qquad (1)$$

$$q_1 = \frac{h_1 - h_2}{R_1} \qquad (2)$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \qquad (3)$$

$$q_2 = \frac{h_2}{R_2} \qquad (4)$$

- Taking the Laplace transform of the 4 equations

$$C_1 s H_1(s) = Q_i(s) - Q_1(s) \qquad (5)$$

$$Q_1(s) = \frac{1}{R_1}\left(H_1(s) - H_2(s)\right) \qquad (6)$$

$$C_2 s H_2(s) = Q_1(s) - Q_2(s) \qquad (7)$$

$$Q_2(s) = \frac{1}{R_2} H_2(s) \qquad (8)$$

Liquid-level System

- By substituting equation (8) in (7)

$$C_2R_2sQ_2(s)+Q_2(s)=Q_1(s) \qquad (9)$$

Substituting equation (9) in (5)

$$C_1sH_1(s)=Q_i(s)-(C_2R_2s+1)Q_2(s) \qquad (10)$$

From equations (6) and (8)

$$\begin{aligned} H_1(s) &= R_1Q_1(s)+H_2(s) \\ &= R_1Q_1(s)+R_2Q_2(s) \end{aligned}$$

From equation (9)

$$H_1(s)=C_2R_1R_2sQ_2(s)+R_1Q_2(s)+R_2Q_2(s) \qquad (11)$$

Liquid-level System

- By substituting equation (11) in (10)

$$C_1C_2R_1R_2s^2Q_2(s)+C_1R_1sQ_2(s)+C_1R_2sQ_2(s)=Q_i(s)-(C_2R_2s+1)Q_2(s)$$

$$Q_2(s)\left(C_1C_2R_1R_2s^2+C_1R_1s+C_1R_2s+C_2R_2s+1\right)=Q_i(s)$$

$$\therefore \frac{Q_2(s)}{Q_i(s)}=\frac{1}{C_1C_2R_1R_2s^2+(C_1R_1+C_1R_2+C_2R_2)s+1}$$